MATH 42-NUMBER THEORY
PROBLEM OF THE DAY #2
DUE THURSDAY, FEBRUARY 3, 2011

Today, you will begin to discover the secrets of the Magic Box!

Here is a first example. First, a continued fraction.

$$\frac{52}{35} = 1 + \cfrac{1}{2 + \cfrac{1}{17}}$$

Next, an "empty" magic box.

| | | 1 | 2 | 17 |
|---|---|---|---|---|
| 0 | 1 | | | |
| 1 | 0 | | | |

Now, the magic box filled.

| | | 1 | 2 | 17 |
|---|---|---|---|---|
| 0 | 1 | 1 | 3 | 52 |
| 1 | 0 | 1 | 2 | 35 |

**1.** Describe as many patterns as you can. Here are some suggestions.

- Compare the entries in the magic box with the convergents of the continued fractions.
- Can you fill in an "empty" magic box without computing convergents? (Hint: The two rows you fill in are independent of each other, so you could fill in the entire first row without doing the bottom row, or the entire bottom row without doing the top row. Find the pattern for each row.)
- What can you say about the $2 \times 2$ determinants? (In the above example, the determinants are $\left|\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right| = 0 \cdot 0 - 1 \cdot 1$, $\left|\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right| = 1 \cdot 1 - 0 \cdot 1$, $\left|\begin{smallmatrix} 1 & 3 \\ 1 & 2 \end{smallmatrix}\right| = 1 \cdot 2 - 3 \cdot 1$, and so on.)
- How might that help you find $x$ and $y$ such that $52x + 35y = 1$? $ax + by$?

To help your magic box quest, here are some more examples of filled magic boxes.

| | | 1 | 2 | 1 | 1 | 3 | 1 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 3 | 4 | 7 | 25 | 32 |
| 1 | 0 | 1 | 2 | 3 | 5 | 18 | 23 |

| | | 2 | 1 | 1 | 1 | 3 |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 5 | 8 | 29 |
| 1 | 0 | 1 | 1 | 2 | 3 | 11 |